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COMMENT

A note on the Berry phase for systems having one degree of freedom

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Abstract. A one-dimensional arbitrary system with quantum Hamiltonian $H(\hat{q}, \hat{p})$ is shown to acquire the 'geometric' phase $\gamma(C) = (1/2\hbar)\oint_C (p_0\,\mathrm{d}q_0 - q_0\,\mathrm{d}p_0)$ under adiabatic transport $q \to q + q + q_0(t)$ and $p \to p + p_0(t)$ along a closed circuit C in the parameter space $(q_0(t), p_0(t))$. The non-vanishing nature of this phase, despite only one degree of freedom (q), is due ultimately to the underlying non-Abelian Weyl group. A physical realisation in which this Berry phase results in a line spread is briefly discussed.

The non-integrable geometric phase $\gamma(C)$ acquired by a quantal system under adiabatic modification of its Hamiltonian along a closed circuit C in the parameter space has been the subject of intense discussion following its recent formulation by Berry [1, 2]. To the best of our knowledge the work reported so far has often involved a parameter space of non-trivial topology. More specifically, most of the discussion is in terms of a two-level system modelled by a pseudospin-$\frac{1}{2}$ in a pseudomagnetic field $\boldsymbol{B}$ and the parameters are the three components of the field. The Berry phase is then half the solid angle subtended by the parameter circuit C at the origin $\boldsymbol{B} = 0$, the point of degeneracy. The non-zero phase is a consequence of the non-Abelian nature of the underlying group SU(2). Inasmuch as a quantum system with one degree of freedom is essentially non-degenerate, the above picture is not applicable in the present case. Nevertheless, in this comment we demonstrate explicitly a non-vanishing Berry phase for an arbitrary ststem having just one degree of freedom and with a two-dimensional parameter space of trivial topology $\mathbb{R}^2$ that comes naturally with it.

Consider an arbitrary system with one degree of freedom described by the quantum Hamiltonian $H(\hat{q}, \hat{p})$. Let the system be modified adiabatically as $q \to q + q_0(t)$, $p \to p_0(t)$, where $q_0(t), p_0(t)$ are slowly varying c numbers and constitute the two-dimensional parameter space (a plane). The modification can be effected by the unitary transformation:

$$U(q_0(t), p_0(t)) = \exp\left(\frac{\mathrm{i}}{\hbar}(p_0(t)\hat{q} - q_0(t)\hat{p})\right). \tag{1}$$

The Berry phase $\gamma_n(C)$ acquired by the adiabatically evolving eigenstate $|\phi_n(q_0(t), p_0(t))\rangle$ is then given by [1]

$$\gamma_n(C) = \mathrm{i}\oint_C \left(\left\langle \phi_n(q_0, p_0)\left|\frac{\partial}{\partial q_0}\right|\phi_n(q_0, p_0)\right\rangle \mathrm{d}q_0 + \left\langle \phi_n(q_0, p_0)\left|\frac{\partial}{\partial p_0}\right|\phi_n(q_0, p_0)\right\rangle \mathrm{d}p_0\right). \tag{2}$$

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Recalling that in the adiabatic limit (i.e. $U^{+}\,\partial U/\partial t \to 0$)

$$|\phi_n(q_0, p_0)\rangle = U(q_0, p_0)|\phi_n(0,0)\rangle \tag{3}$$

and factorising $U(q_0, p_0)$ appropriately as

$$U(q_0, p_0) = \exp\left(-\frac{\mathrm{i}}{\hbar} q_0(t)\hat{p}\right) \exp\left(\frac{\mathrm{i}}{\hbar} p_0(t)\hat{q}\right) \exp\left(\frac{\mathrm{i}}{2\hbar} q_0(t)p_0(t)\right) \ldots \tag{4}$$

we get

$$\gamma_n(C) = \frac{1}{2\hbar}\oint_C (p_0\,\mathrm{d}q_0 - q_0\,\mathrm{d}p_0) = -\left(\frac{\text{circuit area}}{\hbar}\right). \tag{5}$$

This completes the proof of our assertion. We note that this phase is the same for all eigenstates (and therefore for all states) and that it is independent of the nature of the Hamiltonian $H(\hat{q}, \hat{p})$. It is also clear that the non-vanishing of this phase stems from the non-commutativity of $\hat{q}$ and $\hat{p}$ (i.e. the non-Abelian nature of the underlying Weyl group). That the Berry phase is the same for all states implies that the classical Hannay angle [3] is zero. This is also to be expected from the fact that the classical limit of the non-Abelian Weyl group is the Abelian translation group in a plane. For simplicity we have considered a system with one degree of freedom, but now it is clear that, if we had a system with n degrees of freedom and transported the system along a closed curve in the parameter space $(\boldsymbol{q}_0(t), \boldsymbol{p}_0(t))$ which is now a $2n$-dimensional phase space, the Berry phase will be related to the sum of the areas of the projected circuits in the planes $(q_0^{(i)}(t), p_0^{(i)}(t))$, $i = 1, 2, \ldots, n$.

Now we turn to the question of physical realisability of the adiabatic transport $\hat{q} \to \hat{q} + q_0(t)$ and $\hat{p} \to p_0 + \hat{p}(t)$. For a charged particle (electron), $p \to p(t)$ can be effected by an externally applied vector potential $\boldsymbol{A}(t)$. Thus, consider an electron with a Hamiltonian $\hat{p}^2/2m + V(\hat{q})$ placed in a polarised electromagnetic radiation field ('pump'). Also, let the atom be matrix bound (i.e. embedded in a lattice), so that its centre of mass executes an oscillatory nuclear motion, $q_0(t)$. Then we have for the electronic Hamiltonian (in the adiabatic limit)

$$H = \frac{1}{2m}\left(\hat{p} - \frac{e}{c}\boldsymbol{A}(t)\right)^2 + V(\hat{q} - q_0(t)). \tag{6}$$

Here $\boldsymbol{A}(t)$ is the vector potential and $q_0(t)$ is the centre-of-mass nuclear coordinate, all referred to the laboratory frame. As usual, the spatial variation of $\boldsymbol{A}(t)$ over the atomic length scale is ignored (the dipole approximation). Thus we have $\hat{\boldsymbol{p}}(t) = -(e/c)\boldsymbol{A}(t)$. To ensure adiabaticity, the frequencies associated with $\boldsymbol{A}(t)$ and $\boldsymbol{q}_0(t)$ must be kept small compared with the electronic energy level spacings (the Born–Oppenheimer approximation). Now, when the frequency of radiation equals that of the centre-of-mass motion, we will have in general an elliptic circuit in the parameter space and a non-zero Berry phase per cycle. (This picture is, of course, true for any matrix bound atom or molecule, as in a molecular solid, say, or an ion trapped in an RF trap or Paul trap. The latter is a good approximation to a nearly isotropic three-dimensional harmonic potential well for the single ion in question as is well known in the context of laser cooling when the natural line width of electronic transition is much less than the oscillator frequency [4].) It is clear that the number of circuit excursions per unit time is to be interpreted as a level shift for the electronic subsystem. The shift is, of course, the same for all eigenstates and hence undetectable in the absorption spectrum if the electronic subsystem is now probed (one must have a

transition to another *reference* system, not subject to the above shift). On the other hand, for the off-resonance case, i.e. the period of $\boldsymbol{A}(t)$ not equal to that of $\boldsymbol{q}_0(t)$, the phase change may not accumulate on the average (and therefore no level shift), but it can still contribute to a *line spread* due to its asynchronous almost chaotic variation in time. This 'geometric' level width will scale as the square root of the intensity of the 'pump' radiation. Lineshape analysis is possible as in the case of a phase-modulated system.

In conclusion, we have demonstrated the Berry phase for a system with one degree of freedom for which the classical Hannay angle is zero. Its possible manifestation as the linewidth of an electronic transition is noted for a matrix bound atom or an ion trapped in a potential well.

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